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SPHERES, CUBES AND BOXES: GRAPH DIMENSIONALITY AND NETWORK STRUCTURE *

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This is an examination of a set of dimensional conceptions of graphs that might be used to shed light on the structural complexity of social networks. Problems of characterizing various conceptions are explored and computational methods are reviewed.

1. Introduction

Dimensional ideas are common in every branch of science. Social network analysis is no exception. Almost every study of social networks involves discussion of the parameters, factors or dimensions underlying and embodied in the pattern of relations that is being examined. Different words are used, but the idea is the same; we would like to be able to reduce the enormous number of structural characteristics of networks to a relatively small set that are somehow basic or fundamental.

One way to approach this problem is by examining the *dimensionality* of a network of interest. In so doing, one is asking about the structural complexity of the object being studied. Is this, one asks, a simple pattern that varies, say, with respect to a single characteristic. Or is it so complex that it requires taking five or ten or twenty different features into account?

These are the sorts of questions that lead to a concern with dimensionality in social networks research. And these are the sorts of ques-

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tions that motivate the present essay. What I am doing here is examining a set of conceptual tools that can be used to study the dimensionality of social networks. Existing conceptions of dimensionality will be clarified and in some cases modified, and extensions of present models will be proposed.

2. Existing models of graph dimensionality

Social networks are typically represented as graphs. Graph theory, moreover, embodies a standard topological conception of dimensionality due to Veblen (1922). From this perspective a graph is a simplicial complex, the simplices of which are either points or point pairs (lines). Then each simplex in the complex has a dimension equal to one less than the number of points it contains and the dimension of the complex is the maximum dimension of any of its simplices. Thus the null graph has dimension 0 and all other graphs have dimension 1. Obviously this provides a trivial measure of dimensionality from the perspective of the kinds of applications we are considering here.

More recently Freeman (1980) proposed a variant of this perspective based on the algebraic topology of Atkin (1972). This variant can also be derived from the study of hypergraphs (Seidman 1981). In this case, the cliques – the maximum complete subgraphs of a graph – are defined as simplices. A clique of n points is represented as an n-gon, and its dimension is one less than the number of points it contains. The dimension of the entire graph – the simplicial complex – is that of its largest clique. This permits greater differentiation among graphs, but it still misses capturing the sort of intuition that might be useful for applications in social network analysis. We must seek elsewhere for a dimensional idea that can be used profitably in the study of social networks.

The most natural way to think about dimensionality is based on geometry. We draw on spatial analogs and seek a way of capturing the structure of a network in some sort of geometric representation. We map from points and edges in a graph to points and distances in a metric space. In this way we can find the dimensionality of the metric space and, in effect, transfer it to the original graph.

Both Guttman (1977) and Roberts (1969b) have introduced particular dimensional conceptions for the study of graphs. Here a more general treatment will be given.[1] The formalisms of Roberts and Guttman will be extended and related in the context of a more general treatment of graph embeddings.

3. Embedding graphs in metric spaces

Given a finite set consisting of n points,

$$A = \{a, b, \ldots\},$$

and a reflexive, symmetric binary relation, R, on $A \times A$, let G denote the graph $\langle A, R \rangle$. Note that G has a loop at each point.

Each point in A is assigned a real-valued coordinate function f_i, in each dimension of some m dimensional Minkowski r-metric space ${}_rM^m$, so that for each $a \in A$,

$$\langle f_1(a), f_2(a), \ldots, f_m(a) \rangle$$

is a vector locating a in ${}_rM^m$.

Now we seek an appropriate metric in ${}_rM^m$ such that, if possible, for all a, b in A,

$$aRb \leftrightarrow d_{ab}(\langle f_1(a), f_2(a), \ldots, f_m(a) \rangle, \langle f_1(b), f_2(b), \ldots, f_m(b) \rangle) \leq \delta, \tag{1a}$$

where δ is a positive number. Thus, all adjacent pairs in G are placed at most at distance δ from each other in ${}_rM^m$ and all non-adjacent pairs are farther than δ.

It is necessary, however, to choose a metric that satisfies the usual condition for a Minkowski r-metric space:

$$d_{ab} = \left[\sum_{i=1}^{m} |f_i(a) - f_i(b)|^r \right]^{1/r}$$

for all a, b, $\in A$, and where $r \geq 1$.

[1] Although Guttman's paper was published in 1977, the work underlying it was done *circa* 1965 in collaboration with Harary.

Since r may vary, this condition does not uniquely define a metric in ${}_rM^m$, until r is specified; two suggestions have been made.

Roberts (1969b) proposed a "dominance" conception where $r = \infty$ and therefore [2]

$$d_{ab} = \max_{i=1}^{m} |f_i(a) - f_i(b)|. \tag{2a}$$

This suggests that each point in ${}_\infty M^m$ may be seen as located at the center of a square, cube or hypercube with sides of length 2δ parallel to the axes of the space. The distance between two points in ${}_\infty M^m$ is determined by their largest separation in any dimension and they fall within each other's cubes if and only if their corresponding domain points in A are adjacent in G. The intuition for such a formulation is that when two points are separated by a great amount in any dimension that dimension "takes over" and completely dominates their distance. This will be called a Type I embedding.

Guttman (1977) suggested an alternative using the usual Euclidean metric where $r = 2$ and

$$d_{ab} = \left(\sum_{i=1}^{m} [f_i(a) - f_i(b)]^2 \right)^{1/2}. \tag{2b}$$

In this case each point in ${}_2M^m$ can be viewed as located at the center of a circle, sphere or hypersphere with radius δ. The distance between two points is simply their Euclidean distance and they fall within one another's spheres if and only if their corresponding domain points in A are adjacent in G. I shall call this a Type II embedding. [3]

Both Roberts (1969b) and Guttman (1977) suggested alternative versions of their embedding models to be produced by modifying expression (1a) above. Instead of defining δ as a scalar, we can specify a

[2] The parameter, r, in a Minkowski r-metric is, in effect, a weighting parameter. When $r = 1$ all distances are weighted equally. When $r = 2$, they are squared, so that greater distances are weighted more heavily. Setting $r = \infty$ exacerbates those weightings to the degree that all distances less than the greatest one fall out entirely and expression (2a) follows.

[3] The "city-block" metric where $r = 1$ is another obvious candidate for embedding. Although it has been used in cognitive research, I could, however, find no work on embedding graphs in metric spaces with $r = 1$. In a personal communication Jim Lingoes has indicated that, "The cases of $r = 1$ and $r = \infty$ cannot be distinguished under certain orthogonal transformations."

Δ vector by defining a δ_a for each a in A. Thus, each $a \in A$ is still associated with an hypersphere or hypercube, but in this case these objects may differ in size.

Roberts maintained his use of the dominance metric (2a) and proposed that with a Δ vector we let

$$aRb \leftrightarrow d_{ab} \leq \delta_a + \delta_b \qquad (\forall a, b \in A). \tag{1b}$$

This means that the two centers a and b do not need to fall within the boundaries of each other's hypercubes when a and b are adjacent in G. Instead, adjacency in G is associated only with overlapping boundaries; neither hypercube need necessarily contain its adjacent partner's center. [4] Note however, that one hypercube may be completely contained in another given these conditions. This will be called a Type III embedding.

Guttman's proposal maintained his use of the Euclidean metric (2b) and his embedding rule also differed from the one proposed by Roberts. Guttman suggested that with a Δ vector we let

$$aRb \leftrightarrow d_{ab} \leq \min(\delta_a, \delta_b) \qquad (\forall a, b \in A). \tag{1c}$$

Thus, Guttman's points are still the centers of hyperspheres and each sphere must contain the center of any neighbor that is adjacent in G. Here, however, the hyperspheres may vary in size. And, as in the Type III Roberts case, a sphere may be completely contained in another. This is Type IV embedding.

Both Guttman and Roberts proposed one additional condition to constrain the mappings in all four of these cases. Both define graph equivalence of two elements a and b of the set A as

$$a \equiv_G b \leftrightarrow (\langle a, x \rangle \in R \leftrightarrow \langle b, x \rangle \in R) \qquad (\forall x \in A).$$

Then they propose the condition that

$$a \equiv_G b \leftrightarrow f_i(a) - f_i(b) = 0 \qquad (i = 1 \text{ to } r),$$

[4] Roberts (1969b) actually provided a slightly different representation of embedding in the (1b), (2a) metric. Instead of referring to squares, cubes and hypercubes of varying sizes, he specified rectangles and generalized rectangles of varying sizes that he called "boxes." There is, however, no loss of generality in the present formulation.

which means that all points in G that are linked to each other and to exactly the same others must be mapped to a single point in ${}_rM^m$ and that only such graph equivalent points may be so mapped.

We end up then with four different conceptions of the dimensionality of graphs all based on mappings from graphs to metric spaces. Two by Roberts, Types I and III, use the dominance metric, (2a), and two by Guttman, Types II and IV, use metric (2b) and thus produce Euclidean embeddings. Type I by Roberts and Type II by Guttman use rule (1a) and thus define a constant δ that implies a uniform area of "influence" for all points. The other two permit differing values of δ_a to be assigned to individual points, but differ in the embedding rule they use. Roberts uses the distance sum (1b) along with the dominance metric (2a) and produces a Type III embedding. However, Guttman proposes using the minimum distance (1c) along with the Euclidean metric (2b) and suggests a Type IV embedding.

Before we can even think of applying any of these four conceptions of dimensionality we need to know a good deal more about them. In particular, we would like to know first of all whether they can in fact be used to embed graphs in metric spaces.

Roberts (1969b) showed that any graph, G, containing n points can, using either Type I or Type III embedding rules, be embedded in at most n dimensions. And Guttman (1977) proved that, using Type II or Type IV embedding rules, a graph, G, of n points can be embedded in at most $n-1$ dimensions.

Thus, any of the four types of embedding rules can be used to embed any graph. However, it is clear that, in any of these embeddings, the actual locations of points in ${}_rM^m$ are not uniquely determined. Pairs of points are either "close" or "far"; the questions of how close or how far are meaningless.

What *is* meaningful, however, is m, the *minimum* dimensionality of the space in which a given graph can be embedded. Both Guttman and especially Roberts have shown interesting classes of graphs that can be embedded in spaces where m is much less than n.

One factor that can reduce dimensionality in any type of embedding is a consequence of the graph equivalence described above. Roberts (1969b) and Guttman (1977) both showed that given a graph, G, containing two points $a \equiv_G b$, the minimum dimensionality

$$m(G) = m(G-a) = m(G-b).$$

Thus, since a and b are both mapped to the same point in ${}_rM^m$ they cannot make independent contributions to dimensionality.

In every case, embeddings in very small numbers of dimensions are informative. In particular $m = 1$ dimensional embeddings turn out to be interesting since they suggest an intuitive "feel" for what a given embedding rule is all about. Then too, it is useful to know something about the properties of graphs that require minimal maximal dimensionality in the several embeddings. All these properties can help to characterize these embeddings; they can tell us important things about how dimension, in the context of each of these conceptions, is related to structural properties of networks. These questions will be addressed in the next four sections where each of the four embeddings will be examined in turn.

4. Type I embeddings

Type I embeddings use a constant δ (1a) along with a dominance metric (2a). Roberts (1969b) calls this dimensionality *cubicity*, and characterizes its minimal form as an *indifference graph*.

Theorem I.1. (Roberts 1969b). Using a Type I embedding, a graph is embeddable in no more than one dimension [5] if and only if it is an indifference graph.

Indifference graphs organize a set of observations into an ordered sequence. If they are connected, they contain either a single point or an ordered sequence with exactly two extremes. Thus they cannot contain any of the graphs shown in I to IV of Figure 1 as generated subgraphs (Roberts 1969a). [6]

Indifference graphs turn out to be important in the modelling of non-preferences in economics and psychology, and in ordering observa-

[5] Given that both Guttman and Roberts locate all points that are graph equivalent in the same spot, complete graphs – regardless of their size – are all collapsed into a single point in ${}_rM^m$. Given this collapse, all complete graphs are embeddable in ${}_rM^0$. But since such single point graphs are trivially indifference graphs they must be included in this theorem.

[6] Given a graph, $G = \langle A, R\rangle$, $G' = \langle A', R'\rangle$ is a generated subgraph of G if and only if $A' \subseteq A$ and for all $a, b \in A'$, $aRb \in R' \leftrightarrow aRb \in R$. Thus, any pair of points that are connected in the original graph must remain connected in the generated subgraph.

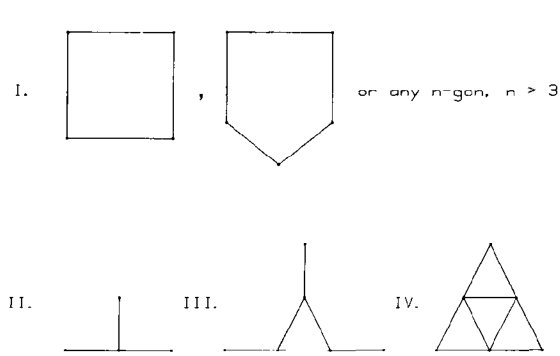

Figure 1.

tions in archaeology, developmental psychology and political science (Roberts 1978). They are also potentially useful in modelling the sorts of role similarity structures White and Reitz (1981) have described.

Roberts has shown a number of important properties of Type I embeddings as the number of dimensions increases from 1. In particular, we can see that higher dimensional embeddings are generalizations of embeddings in one dimension by the next theorem.

Theorem I.2. (Roberts 1969b). Using a Type I embedding, a graph that is embeddable in ${}_{\infty}M^m$ $(m \geq 1)$ is the intersection of m indifference graphs. [7]

Roberts went on to show the conditions under which a graph of n points exhibits the maximum dimensionality. A complete p-partite graph, $K(n_1, n_2, \ldots, n_p)$ is made up of p classes, containing $n_1, n_2, \ldots, n_p$ $(n_i > 0)$ points respectively, such that no pairs of points that both fall within a class are adjacent and all pairs of points that fall in different classes are adjacent. Thus, *stars* $S(n_i)$, are complete 2-partite graphs $K(1, n-1)$ where one point, the center, is adjacent to $n_i = n - 1$ other points, none of which is adjacent to any of the others.

Roberts (1969b) showed that as n_i increases, stars require increasing dimensionality for embedding. Thus, $S(n_i)$ is embeddable in m-space if

[7] This intersection idea is straightforward. A two-dimensional Type I embedding, for example, yields two sets of indifferences each with its own pattern of overlap. Only those pairs of points that overlap on *both* dimensions represent adjacencies on the domain graph.

and only if $n_i \leq 2^m$, so the minimum dimensionality, $m(S(n_i)) = [\log_2(2n_i - 1)]$ where $[x]$ is the greatest integer in x.

The dimensionality of any complete p-partite graph, then, can be expressed as the sum of the dimensionalities of the stars that make it up: Let

$$G = K(n_1, n_2, \ldots, n_p).$$

Then if $p = 1$,

$$m(G) = \begin{cases} 1 & \text{if } n_p > 1, \\ 0 & \text{if } n_p = 1; \end{cases}$$

if $p > 1$

$$m(G) = \sum_{j=1}^{p} m(S(n_j)).$$

Roberts proved that, given n points, the maximum dimensionality, $d(n)$, will be achieved only in a complete p-partite graph and only under rather remarkable conditions:

Theorem I.3. (Roberts 1969b). Using a Type I embedding, an n-point graph has maximum dimensionality, $d(n)$, if and only if

(a) it is a complete p-partite graph, $K(n_1, n_2, \ldots, n_p)$, and
(b) as many of the n_i as possible are set equal to 3 with the remaining n_i set at 2 or 1 as necessary.

Thus, $d(n) = [\frac{2}{3}n]$ if $n \neq 3$ and $d(3) = 1$.

Finally, a word of caution must be introduced concerning these embeddings and the intuitive notion of order. We can see from the discussion of graph equivalence above that we are embedding, not graphs, but their blockmodelled images. All points that are graph equivalent are reduced to a single point, so we are, in effect, embedding reduced graphs.

Now if we think about the sort of ordered sequence underlying each dimension of a Type I embedding, and we remember that we are working with a reduced graph, it is clear that each of the m dimensions is represented as an ordered sequence. If the reduced graph is connected, each dimension may be ordered in exactly two ways, one being the complete reversal of the other. But this is true if and only if the reduced graph is connected! If it is unconnected, each connected component may be ordered in two ways on each dimension and each element in each pair of components may arbitrarily either precede or follow the other in each dimension. Thus, the possible orders grow very rapidly even with a few components. In general, then, the orderings provided by this embedding are probably only directly interpretable when they are used to embed connected reduced graphs.

5. Type II embeddings

Type II embeddings use a constant δ (1a) with an ordinary Euclidean metric (2b). Guttman (1977) referred to this embedding simply as "dimension," and he did not characterize its minimal form. However, since metric (2b) and metric (2a) are identical in ${}_rM^1$, such a characterization is direct.

Theorem II.1. Using a Type II embedding, a graph is embeddable in no more than one dimension if and only if it is an indifference graph.

Although he gave illustrations of two-dimensional cases, Guttman (1977) also neglected to characterize higher dimensional embeddings. We have, therefore, no theorem for a Type II embedding that corresponds to I.2. We shall see, however, that an m-dimensional Type II embedding is certainly *not* the intersection of m indifference graphs. What it is is unknown.

The dimensionality of Type II embeddings, like that of Type I, does grow with increasing n_i in stars, $S(n_i)$. It seems, however to grow at a different rate than that for Type I. Roberts provided an exact method for calculating the value of m for each value of n_i in a Type I embedding. Although the corresponding problem has been studied for hundreds of years, no simple general function has been discovered for the Euclidean space of Type II.

The corresponding problem in Euclidean space is called "sphere packing." The question is, given an m-dimensional space, how many unit discs, spheres or hyperspheres can be packed around a unit reference disk, sphere or hypersphere so that those surrounding objects are all in contact with the reference object.

For the present purpose we must add one additional restriction: the surrounding objects not only must touch the reference sphere, they must not touch each other. If two of them did touch, their centers would be as close to one another as they were to the reference center, and this would violate the embedding rule (1a).

In any case, if we knew the maximum sphere packing, we could – with a little juggling – solve the relation of m to n_i. Unfortunately, except for a very few cases, we do not. We do, however, know the lower bounds for quite a few cases, and those allow us to compare some Type I and II embeddings of $S(n_i)$.

Some results are shown in Table 1. In lower dimensions Type II embeddings increase in dimensionality more slowly than those of Type I. However, for dimension 8, Type I embeddings can include a star, $S(n_i)$, where $n_i \leq 256$. But a Type II embedding can only take a star of

Table 1
Some maximum n_i for embedding $S(n_i)$ in dimensions of Type I and II.

Minimum dimension m	Type I maximum n_i for embedding $S(n_i)$	Type II lower bound of maximum n_i for embedding $S(n_i)$	Source for theorem and comments
1	2	$=$ 2	(Obvious)
2	4	$=$ 5	Thue (1910) proved that 6 may be packed but touch each other
3	8	$=$ 12	Coexter (1963) proved that 12 need not touch each other.
4	16	$\geq$ 24	Minkowski (1905) *
5	32	$\geq$ 40	
6	64	$\geq$ 72	Blichfeldt (1935) *
7	128	$\geq$ 126	
8	256	$=$ 240	Odlzyko and Sloane (1979) *
9	512	$\geq$ 306	Leech and Sloane (1970) *
⋮	⋮	⋮	

* These limits refer to sphere packings; it is not known whether the packed peripheral spheres are or are not touching each other. To be safe, we could subtract 1 from the values in column three in appropriate cases.

$n_i \leq 240$.[8] About all that can be said at this point is that stars with growing n_i require increasing dimensionality in both Type I and Type II embeddings and that the rates of increase differ between the two.

Guttman (1977) made no attempt to show the maximum dimensionality graph of size n, and therefore provided nothing parallel to Theorem 1.3. Thus, current knowledge about Type II embeddings is seriously limited. Much more needs to be known before embeddings can be fruitfully used.

All of the cautions suggested above with respect to interpreting the dimensions of type I embeddings also apply to those of Type II. Moreover, the fact that the Euclidean embeddings of Type II can be rotated, suggests, at first glance, that these might add an additional arbitrary element in the orderings of points along each dimension. This suggestion is wrong, however. Rotation along any dimension simply results in the uniform overall shrinkage of all interpoint distances along that dimension until they collapse to a point, followed by their expansion until they are spaced out as before but in the reverse order. Thus, since δ may be varied, rotation has no effect on ordering except at the unique point where all interpoint distances are uniformly zero.

6. Type III embeddings

Type III embeddings use a Δ vector with an embedding rule requiring only overlaps (1b), and like those of Type I, they are made in the context of the dominance metric (2a). Roberts (1969b) introduced this embedding rule and called it *boxicity*. He showed that it was a realization of a Type III embedding and characterized its minimal form as an *interval graph*.

Theorem III.1 (Roberts 1969b). Using a Type III embedding, a graph is embeddable in no more than one dimension, if and only if it is an interval graph.

[8] Or perhaps $n \leq 239$ depending on whether the packing model permits the peripheral hyperspheres to touch the center one without touching each other. If it does not, taking 1 away will solve the problem since the others can also be shifted around the center slightly so that they no longer are in contact with each other.

Interval graphs are well known (Lekkerkerker and Boland 1962; Gilmore and Hoffman, 1964; Fulkerson and Gross, 1965). They consist of sets of linearly ordered elements; they can be mapped to overlaps of intervals of the real number line. A graph is an interval graph if and only if it is a rigid circuit graph (excludes I in Figure 1) and it has no asteroidal triple (thus excluding III and IV but not II of Figure 1).

Thus, an interval graph has the characteristics of an indifference graph except that the requirement for equal intervals is relaxed. [9] This means that in an interval graph one interval may be completely contained in another. These kinds of graphs have important applications in biology and archaeology, and have considerable potential in social networks, for example, in the study of co-citation networks in the sociology of science. The question of dimension in that context involves looking at whether the underlying structure is linear or "splits off" into different lines of inquiry.

Again, as in the earlier Type I case, Roberts (1969b) shows that higher dimensional embeddings are a generalization of one-dimensional ones:

Theorem III.2. (Roberts 1969b). Using a Type III embedding, a graph that is embeddable in ${}_{\infty}M^{m}$ $(m \geq 1)$ is the intersection of m interval graphs.

Similarly, Roberts developed the conditions for maximum dimensionality. He showed that under these conditions, where one cube may completely contain another,

$$m(S(n_i)) = 1 \leftrightarrow n_i > 1.$$

Moreover,

$$m\big(K(n_1, n_2, \ldots, n_p)\big) = \sum_{j=1}^{p} m\big(S(n_j)\big),$$

so the minimum dimensionality of a complete p-partite graph is equal to the number of partitions that contain more than one element.

[9] As a matter of fact one can characterize an indifference graph as an *equal interval graph*. It is, in effect, a mapping from a graph to a set of equal intervals on the real number line. Two intervals, a and b, overlap each other's centers if and only if aRb in the graph.

Finally, Roberts proved that, for n points, the maximum dimensionality will again be achieved only in a complete p-partite graph.

Theorem III.3. (Roberts 1969b). Using a Type III embedding, an n-point graph has maximum dimensionality, $d(n) = [n/2]$, for all n.

Because one cube can completely contain another in Type III embeddings, the problem of interpreting dimensions in terms of ordered sequences is even more difficult here. All the problems pertaining to Type I embeddings still hold, plus the fact that if several small cubes are contained in a single large one, they may be placed in any absolutely arbitrary order in each dimension. Thus, difficulties in interpreting dimensions are exacerbated by containments. Users of this model must be very careful not to overinterpret the orders yielded in each dimension of an embedding.

7. Type IV embeddings

Type IV embeddings also use a Δ vector, but use an embedding rule that still requires the centers of two adjacent points to be included in each other's spheres (1c). Furthermore, like Type II embeddings, they do this in the context of a Euclidean space (2b). In this case, however, one sphere may completely contain another.

Guttman (1977) did not effectively characterize these embeddings. However, Seidman has provided a conjecture. [10]

Conjecture IV.1. Using a Type IV embedding, a triangulable graph is embeddable in no more than one dimension.

A triangulable graph is one in which each cycle of length of at least four has a chord. Seidman has shown that such cycles are embeddable in one dimension. The rest of the proof, however, turns out to be more difficult.

Moreover, I can show that an m-dimensional Type IV embedded

[10] Personal communication.

graph is *not* the intersection of m interval graphs.[11] Exactly what it is is more difficult to discover.

It is the case that here, as in the Type III embedding,

$$m(S(n_i)) = 1 \leftrightarrow n_i > 1,$$

but no extension of this result to complete p-partite graphs has been established. Thus, the upper bound for the dimensionality of n points is unknown, and, although Guttman (1977) defined this Type IV embedding, so far we know very little about its properties.

8. Computation

Perhaps Guttman's (1977) greatest contribution to the problem of dimensionality stems from his comments on computing. He showed that at least two of these embeddings, Types II and IV are computationally tractable using available methods. The Guttman–Lingoes Multidimensional Scaling Programs (Lingoes 1973) may be directly used to solve for m in a Type II embedding. The value of m is determined whenever stress is reduced exactly to 0. As a matter of fact, this result provides a new perspective on MDS; Type II embeddings are precisely what MDS is calculating when a binary data matrix is entered.

MDS may be simply adapted to solve for m in Type IV embeddings as well (Lingoes 1982). And adaptations for solving Type I and Type III embeddings are also possible, but, as De Leeuw and Heiser (1977) suggest "unpleasant computational problems" may result. Shepard (1974) showed that the the problem is that existing MDS methods can

[11] Note that a Type III embedding in m dimensions *is* the intersection of m interval graphs. Consider now the class of complete p-partite graphs where n_i for each partition is equal to 2. For Type III, the minimum dimensionality is equal to the number, p, of such partitions. However, for Type IV, we can embed a graph with p as large as we like in exactly 2 dimensions. This embedding simply requires that we place the pair of points that fall *within* each partition on opposite sides of a circle with diameter equal $2\delta + \gamma$ where $\gamma > 0$. Then each and every point may be assigned a constant radius, δ. Now consider two points a and b that fall in the same partition. Thus b must fall at a distance greater than δ (by γ) from point a. But, by setting γ to a vanishingly small value, all other points anywhere in the circle must fall within the boundary of a's circle. Thus, only b is outside the circle of a. This may be done for any number of pairs, and since for Type III $m = p$ while for Type IV $m = 2$, a Type IV embedding is *not* the intersection of m interval graphs.

all get trapped in local minima – where points are locked into a reverse order – particularly in the case of the dominance metric.

Arnold (1971) developed a procedure designed to reduce this problem by beginning with a Euclidean solution where $r = 2$ and then using that solution as the start for a new one with r slightly increased. This process of using the solution of one analysis as the start of the next can be repeated until r is large enough to be a satisfactory approximation to infinity. Carrol and Arabie (1980), however, report some unpublished work of Arabie's that raises doubts about using Arnold's approach when m is small. In any case, some computational methods are already available, but more work remains to be done.

9. Conclusion

Four potentially useful conceptions of the dimensionality of graphs have been shown in the context of a common perspective for examining them. Any graph – it turns out – can be embedded in a Minkowski r-metric space as *either* an *hypersphere* or *hypercube*. There are four models for embedding symmetric graphs. Two, I and III, have been characterized well, but the properties of the other two, II and IV, need elaboration.

Computational procedures, using standard multidimensional scaling algorithms have been shown to be useful for some of these embeddings. Problems with others, have been described. Many of the four dimensional conceptions defined here need further mathematical development. All need to be investigated in terms of their applications in studying networks. Only further mathematical developments can spell out enough of their implications to let social scientists know which of these models can be applied to what problems. Only more applications work can tell mathematicians which models are worth a serious development effort.

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