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# A note on regular colorings of two mode data

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This note follows an earlier suggestion by Borgatti (1989), and Everett and Borgatti (1992a). Here, we extend their notion of regular colorings to bipartite graphs.

In a recent paper, Everett and Borgatti (1992b) generalized their earlier (1991) definition of regular (or role) coloring of graphs. They extended their previous results to demonstrate that colorings can be used to find regular partitionings in digraphs, networks and hypergraphs. The purpose of this note is to suggest an alternative to their hypergraph representation – one that requires only familiar graph theoretic concepts and that lends itself to clear visual displays.

In their earlier paper, Everett and Borgatti (1991) defined a graph $G=(V, E)$ in which the vertices $v \in V$ are social actors and the edges $e \in E$ refer to a symmetric social relation linking pairs of actors. The *neighborhood* of a vertex $N(v)$ is defined as the set of all points that are adjacent to $v$. Each vertex $v \in V$ is assigned a *color* $C(v)$. Then any subset of vertices $S \subseteq P$ can be characterized by its *spectrum* $C(S)$, the set of all the colors assigned to its members.

The assignment of colors to vertices partitions the actors into equivalence sets. Everett and Borgatti showed that these equivalence sets are *regular* (White and Reitz 1983) when, for all $v, w \in V$,

$$C(v)=C(w) \Rightarrow C(N(v))=C(N(w)), \tag{1}$$

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when vertices with the same color are embedded in neighborhoods with the same spectrum.

In their latest paper, Everett and Borgatti (1992b) generalize this result to a two mode, actor by event, data set (Wasserman and Faust 1993). This kind of two mode data begins with a collection of social events. Then a set of social actors are defined by their participation in at least one of the events.

Everett and Borgatti represented this kind of two mode data set as a *hypergraph* $H=(V, E)$. In a hypergraph the vertices $v \in V$ are still social actors, but the edges $e \in E$ now encircle subsets of vertices that are defined by the events. Thus, each edge in $E$ specifies some non-empty subset of actors – those who participated together in one of the events.

For hypergraphs, the notion of neighborhood must be modified. Since each vertex is encircled by one or more edges, we can define the *edge neighborhood* of a vertex $N^{e}(v)$ as the set of edges that encircle it, and the *vertex neighborhood* of an edge $N^{v}(e)$ as the set of vertices it encircles. And since we are dealing with two distinct sets (actors and events), we must define two distinct and non-overlapping color spectra, $C^{v}$ for vertices and $C^{e}$ for edges. Then we must color each vertex $C^{v}(v)$ from one spectrum and each edge $C^{e}(e)$ from the other.

Everett and Borgatti showed that these dual colorings are regular when, for all vertices $v, w \in V$ and all edges $e, f \in E$, two conditions are met:

$$C^{v}(v)=C^{v}(w) \Rightarrow C^{e}(N^{e}(v))=C^{e}(N^{e}(w)), \tag{2}$$

and

$$C^{e}(e)=C^{e}(f) \Rightarrow c^{v}(N^{v}(e))=C^{v}(N^{v}(f)). \tag{3}$$

Thus, a coloring of a hypergraph is regular only when (1) any pair of vertices that have the same color have edge neighborhoods with the same spectrum and (2) any pair of edges that have the same color have vertex neighborhoods with the same spectrum. Vertices of the same color that meet these conditions form a regular equivalence class of actors. Similarly, edges of the same color that meet these conditions form a regular equivalence class of events.

All this is perfectly correct, but because it involves the notion of hypergraphs, it is limited in its potential for application in the study of social networks. In the first place, the hypergraph formalism has not been widely used in network research. But, more important, it does not lend itself to satisfactory visual displays (Freeman 1993). An alternative, one that relies only on ordinary graph theory and does provide good visual images, is based on the notion of *bipartite graphs*.

Any two mode data can be represented as a bipartite graph. A bipartite graph is an ordinary graph $G(V, E)$ in which it is possible to partition its vertices $V$ into two subsets $V_1$ and $V_2$ such that every edge in $E$ joins an element in $V_1$ with an element in $V_2$. In the present context, the elements of $V_1$ are actors and the elements of $V_2$ are social events. Thus, the edges $E$ of the bipartite graph represent the relation of participation that links actors to the events in which they participated.

As Borgatti (1989: 62) and Everett and Borgatti (1992a) have indicated earlier, regularity can also be defined for bipartite graphs. To do so, we can draw directly on Everett and Borgatti's conditions (2) and (3) above. Here the vertices $v_1, w_1, \ldots$ in $V_1$ correspond to their vertices $V$. And those $v_2, w_2, \ldots$ in $V_2$ correspond to their hypergraph edges $E$.

In order to maintain the distinction between these two subsets of vertices, we must color vertices in $V_1$ and $V_2$ with colors drawn from distinct non-overlapping spectra. Then, given this restriction, the vertex coloring is regular when, for all $v, w \in V$,

$$C(v) = C(w) \Rightarrow C(N(v)) = C(N(w)), \tag{4}$$

which is the same simple expression as (1) above. Thus, elements of either set that have the same color are regularly equivalent when they are adjacent to others with the same spectrum. Actors are regularly equivalent when they participate in events that are regularly equivalent. And events are regularly equivalent when they involve the participation of actors who are regularly equivalent.

As an example, Everett and Borgatti presented a subset of the data collected by Davis et al. (1941). The data are records of the participation of 15 women in 14 informal social events; they are shown in Table 1.

Everett and Borgatti's presentation of regularly equivalent subsets of actors and events was restricted by their use of the hypergraph

Table 1
Subset of the Davis *et al.* data

| Actor | Social event | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | a | b | c | d | e | f | g | h | i | j | k | l | m | n |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| 2 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| 4 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 0 |
| 11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 0 |
| 12 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 |
| 13 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 |
| 14 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 15 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 |
| 16 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 0 |

model. They could present only the blocked matrix shown in Table 1. But the same equivalence sets are immediate from an inspection of colored the bipartite graph of Figure 1. This graph can be reduced to produce the color image graph of Figure 2 in which the regularity is apparent.

In summary, what we are suggesting is simply an alternative model for exploring the question of coloring in the case of two mode network data. We believe that the use of the bipartite representation makes it

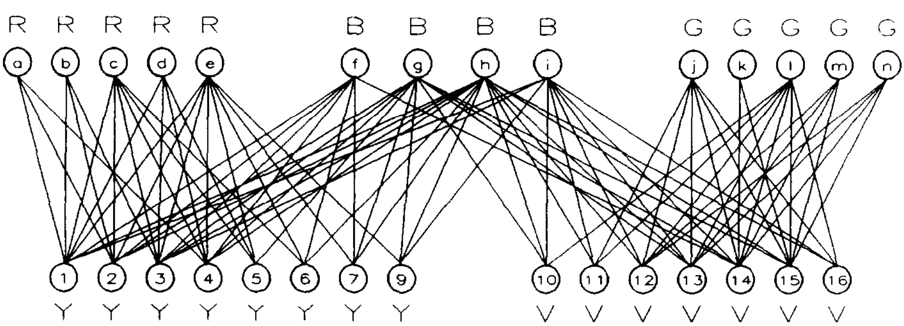


Fig. 1. Subset of the Davis et al. data as a colored bipartite graph.

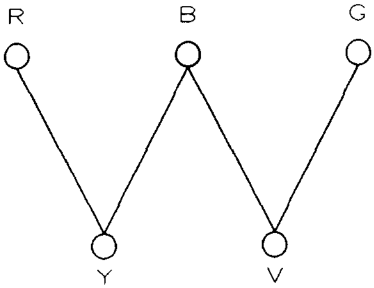

Fig. 2. Image graph of the graph of Fig. 1.

easier to grasp the ideas involved in equivalences in two mode data. Furthermore, we see a distinct advantage in the ability of bipartite graphs to provide the kind of pictorial images of colorings shown here.

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