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Cliques, Galois lattices, and the structure of human social groups [1]

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Abstract

The mathematical definition of clique has never been entirely satisfactory when it comes to providing a procedure for defining human social groups. This paper shows how the Galois structure of containment among cliques and actors can be used to produce an intuitively appealing characterization of groups—one that is consistent with ethnographic descriptions. Two examples, using 'classical' social network data sets, are provided.

1. Introduction

From the outset, sociologists have pretty much agreed that one of their primary concerns is with human groups. They were, and still are, particularly interested in the relatively small, often face-to-face, groups that Tönnies (1887/1940) called *gemeinschaft* and Cooley (1909/1962) called primary. According to tradition, such groups have three main features: (1) they are collections of individuals who are linked to one another by regular interaction and by sentimental ties, (2) they are more or less bounded; they show little if any overlap, and (3) they are internally differentiated; some of their members are more involved with the group than are others (Freeman, 1992).

For the most part, this notion of group has been used in the field in an informal intuitive, or 'sensitizing', way. Ethnographers and participant observers seem to have had little trouble in 'seeing' groups and in assigning members to them. But

[1] Another version of this paper has appeared in French in the Bulletin de Methodologie Sociologique (Freeman, 1993).

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systematic attempts to develop formal models of group structure, and thus to define them in precise terms, have found the going tough.

Attempts to model group structure in formal terms began in the 1940s and continue still. But none of the models proposed so far is entirely successful in capturing the intuitive notion of groups. All the models can be used to find group-like structures, but the structures that they find simply do not match groups as reported by ethnographers.

The aim of the present paper is to propose a new solution to the problem of defining groups. I will draw upon the earliest group model, the *clique*, and show that group structure is displayed, not in the cliques themselves, but in the patterning of their overlap. And that patterning is revealed by organizing cliques into a *Galois lattice*. I will show that when a collection of cliques is recast as a Galois lattice, the lattice reveals precisely the kind of interpersonal patterning that is reported in ethnographic accounts of group structure.

In Section 2 below I will review the clique model. Following that, in Section 3, I will outline some elementary principles of Galois lattices. Then, in Section 4, I will show how the two formalisms can be tied together to reveal group structure. And finally, in Sections 5 and 6, I will show how the results of this clique-lattice analysis produce results that are consistent with ethnographic reports.

2. Cliques and social groups

A formal definition of cliques was simultaneously introduced by Luce and Perry (1949) and by Festinger (1949). This definition was proposed (Luce and Perry, 1949) in order "to determine group structure" in a way that was "both faster and more certain than less systematic methods". Thus, it was intended to provide a formal model for existing intuitive ideas about the organization of groups.

The Luce–Perry definition of the clique begins by designating a set of actors A along with a symmetric binary social relation R in $A \times A$ that links pairs of actors. Then a clique C is a *maximal* subset containing three or more actors among whom all pairs are linked by R. The term 'maximal' in this context simply means that no clique may be contained in a larger clique.

This formal definition of cliques captured the imagination of social scientists right from the beginning. It can be applied to data that recorded either interaction or sentimental ties linking pairs of individuals, and it does specify subsets of individuals who are densely interlinked.

Although the clique definition has intuitive appeal, from the outset it was apparent that it failed when it came to representing group structure as it was conceived by sociologists.

The cliques that are uncovered from actual data on human relationships typically fail to reflect intuitive group structure in four ways: (1) The cliques that are found are too small to satisfy intuition. (2) There are usually too many cliques to satisfy intuition. (3) The cliques that are found often overlap to a degree that violates intuition. (4) Since the clique definition requires that each clique member

Table 1
Data on playing games in the Bank Wiring Room

	I1	I3	W1	W2	W3	W4	W5	W6	W7	W8	W9	S1	S2	S4
I1	0	0	1	1	1	1	0	0	0	0	0	0	0	0
I3	0	0	0	0	0	0	0	0	0	0	0	0	0	0
W1	1	0	0	1	1	1	1	0	0	0	0	1	0	0
W2	1	0	1	0	1	1	0	0	0	0	0	1	0	0
W3	1	0	1	1	0	1	1	0	0	0	0	1	0	0
W4	1	0	1	1	1	0	1	0	0	0	0	1	0	0
W5	0	0	1	0	1	1	0	0	1	0	0	1	0	0
W6	0	0	0	0	0	0	0	0	1	1	1	0	0	0
W7	0	0	0	0	0	0	1	1	0	1	1	0	0	1
W8	0	0	0	0	0	0	0	1	1	0	1	0	0	1
W9	0	0	0	0	0	0	0	1	1	1	0	0	0	1
S1	0	0	1	1	1	1	1	0	0	0	0	0	0	0
S2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S4	0	0	0	0	0	0	0	0	1	1	1	0	0	0

be linked directly to every other clique member, their form hides the important 'internal structure' that social groups are assumed to possess.

As an example, consider a data set that was collected by Roethlisberger and Dixon (1939) among some employees in a Western Electric Company factory. Subjects were 14 men who worked in a 'Bank Wiring Room'. Their job was to produce switchboard banks. Two of these men were inspectors (I1 and I3), three were solderers (S1, S2 and S4) and the remaining nine were wirers (W1 through W9).

An observer recorded data on several different kinds of social relationships that linked these men together. Among the data collected were records of which pairs of men played various games and engaged in horseplay with each other. These data produced the symmetric binary matrix shown in Table 1. There, only those pairs of individuals who had been observed playing games together were assigned a value of 1 in their cell in the matrix; others were tabulated as 0.

Ethnographically, Roethlisberger and Dixon (1939) described these men as divided up into two distinct groups. Group A contained W1, W2, W3, W4, S1 and I1. They described W3 as the 'leader' of Group A (p. 465) and W2 as a marginal member. Group B included W6, W7, W8, W9 and S4. W6, they said (p. 509) was "not entirely accepted by the group" and S4 was "socially regarded as inferior" (p. 483). The other three, W5, S2 and I3, they characterized as "outside either Group A or Group B" (p. 510).

All the cliques contained in the data of Table 1 are shown in Table 2. They illustrate each of the problems with clique analysis listed above: (1) The cliques are all smaller than the groups; the largest clique, for example, contains only five individuals, while Roethlisberger and Dixon's Group A has six members. (2) There are five cliques but only two groups, so there is no way to choose which combination of cliques best represents the group structure. (3) Many of the cliques overlap, while the groups do not; (C1 and C3, for example, share 4 of their 5 members). (4)

Table 2
Cliques formed by game playing in the Bank Wiring Room

	C1	C2	C3	C4	C5
I1	1	0	0	0	0
I3	0	0	0	0	0
W1	1	1	1	0	0
W2	1	1	0	0	0
W3	1	1	1	0	0
W4	1	1	1	0	0
W5	0	0	1	0	0
W6	0	0	0	1	0
W7	0	0	0	1	1
W8	0	0	0	1	1
W9	0	0	0	1	1
S1	0	1	1	0	0
S2	0	0	0	0	0
S4	0	0	0	0	1

If we do select a pair of these cliques (say C2 and C5) as an approximation of the structure of the two groups, we are unable to distinguish between group leaders (like W3), and peripheral members (like W2).

For one or another of these reasons, then, most analysts concluded that cliques were unsatisfactory as a way to characterize the structure of groups (Luce 1950; Alba, 1973; Peay, 1976; Alba and Moore, 1978; Seidman and Foster, 1978; Mokken, 1979; Doreian, 1982; King and Nakornchai, 1982; Seidman, 1983; Yan, 1988; Arabie and Carroll, 1989). Nevertheless, the clique idea has continued to interest investigators. Over the years, its mathematical simplicity and intuitive appeal have kept bringing investigators back to it, but its tendency to uncover groups that do not match observers' descriptions keeps driving them away again.

As I suggested above, one solution to these problems with cliques rests in organizing them in terms of Galois lattices. In the next section, therefore, I will review some of the elementary principles of Galois lattices.

3. Galois lattices

Consider a finite non-empty set $X = (x, y, z, \dots)$ along with a binary relation $\leq$ in $X \times X$. We take the relation $\leq$ to be reflexive, antisymmetric and transitive and the set X, therefore, is *partially ordered*.

Given a pair of elements x and y in a partially ordered set, a *lower bound* of x and y is an element m such that $m \leq x$ and $m \leq y$. A lower bound m is the greatest lower bound, or *meet* when there is no other element b such that $b \leq x$ and $b \leq y$ and $m \leq b$. Similarly, an *upper bound* j is an element such that $x \leq j$, $y \leq j$. If there is no element b such that $x \leq b$, $y \leq b$, and $b \leq j$, j is the least upper bound, or *join*.

Any partially ordered set in which every pair of elements has both a meet and a join is a *lattice*. Consider, for example, the *power set* $P(X)$ of a set X. $P(X)$ contains of all the subsets of X including X itself and the null set $\emptyset$. The elements of the power set form a partial order, based on inclusion, where, if S, $S' \subseteq X$, then

$$S \cap S' \subseteq S,\ S' \text{ and } S,\ S' \subseteq S \cap S',$$

and $S \cap S'$, $S \cup S'$ are the meet and join of S and S'.

Now consider a triple (A, C, M) where A and C are finite non-empty sets and $M \subseteq A \times C$ is a binary relation. The relation M can be used to define a mapping $\uparrow: B \mapsto B\uparrow$ from $P(A)$ to $P(C)$:

$$B\uparrow = \{c \in C \mid (a, c) \in M \quad \text{for all} \quad a \in B\}.$$

Similarly, M can be used to define another mapping $\downarrow: D \mapsto D\downarrow$ from $P(C)$ to $P(A)$:

$$D\downarrow = \{a \in A \mid (a, c) \in M \quad \text{for all} \quad c \in D\}.$$

Let $S(A) = \{A_1\uparrow, A_2\uparrow, \ldots\}$, the collection of images of $\uparrow$, and $S(C) = \{C_1\downarrow, C_2\downarrow, \ldots\}$, the collection of images of $\downarrow$. Since the two mappings, $\uparrow$ and $\downarrow$, are both constructed from the same pairs in the relation M they are inverses of one another. In fact, the subscripts on these subsets can be assigned in such a way that $A_i\uparrow = C_i$ and $C_i\downarrow = A_i$ for all subscripts i.

The subsets that make up $S(C)$ form a lattice under inclusion, as do the subsets that make up $S(A)$. These two lattices are dual inverse,

$$(C_i\downarrow \subseteq C_j\downarrow \Leftrightarrow A_i\uparrow \supseteq A_j\uparrow),$$

and they can be represented in a single lattice $S(C) \times S(A)$ in which each element is identified with a subset in $S(C)$ *and* with a subset in $S(A)$. Thus,

$$(C_i\downarrow, A_i\uparrow) \le (C_j\downarrow, A_j\uparrow) \Leftrightarrow C_i\downarrow \subseteq C_j\downarrow \ \&\ A_i\uparrow \supseteq A_j\uparrow,$$

and an element $(C_i\downarrow, A_i\uparrow)$ of this dual lattice is a lower bound of another $(C_j\downarrow, A_j\uparrow)$ when $C_i\downarrow$ is contained in $C_j\downarrow$, or equivalently, when $A_j\uparrow$ contains $A_i\uparrow$. A dual lattice of this sort, where each element is a pair, is called *Galois*.

A complete Galois lattice may be displayed pictorially by a labeled line diagram. In such a diagram, each element is represented as a point and points are linked by ascending and descending lines to show their ordering. Each point is assigned two labels, one indicating the subset of elements in A it includes and the other indicating the subset of elements in C it includes.

These, then, are the elementary principles of Galois lattices. They were introduced by Birkhoff (1940) and expanded first by Barbut and Monjardet (1970) and later by Wille (1982, 1984, 1985, 1987, 1989) and by Duquenne (1987). Here, concern is focused on the application of these ideas to the analysis of cliques. That issue will be examined in the next section.

4. Galois lattices of network cliques

The application of Galois lattices to cliques is straightforward. The form of the clique data as it was described above is precisely that required to define a Galois lattice. It consists of a triple (A, C, M) in which A is a set of human actors, C is a collection of cliques and M is a binary relation in $A \times C$. In this case, $(a, c) \in M$ may be read 'actor a is a member of clique c'.

To illustrate, let us return to the cliques we found in Roethlisberger and Dixon's (1939) Bank Wiring Room data. Fig. 1 is a Galois lattice that displays the structure of both actors and cliques. Note that, since actors I3 and S2 were not involved in playing games, the lattice display includes only those 12 individuals who were organized into five cliques of game players.

All of the paired subset elements of $S(A)$ and $S(C)$ can be seen in the figure. The top-most point represents the pair of subsets containing the set of all 12 individuals and no cliques at all. The bottom-most point represents the pair containing the set of all five cliques and no individuals. Intermediate points are labeled to show which subsets of cliques and which subsets of individuals they represent.

In practice, it is usually easier to make sense of a lattice in which the labeling is reduced. Each point is labeled only with the names of actors for which it is the

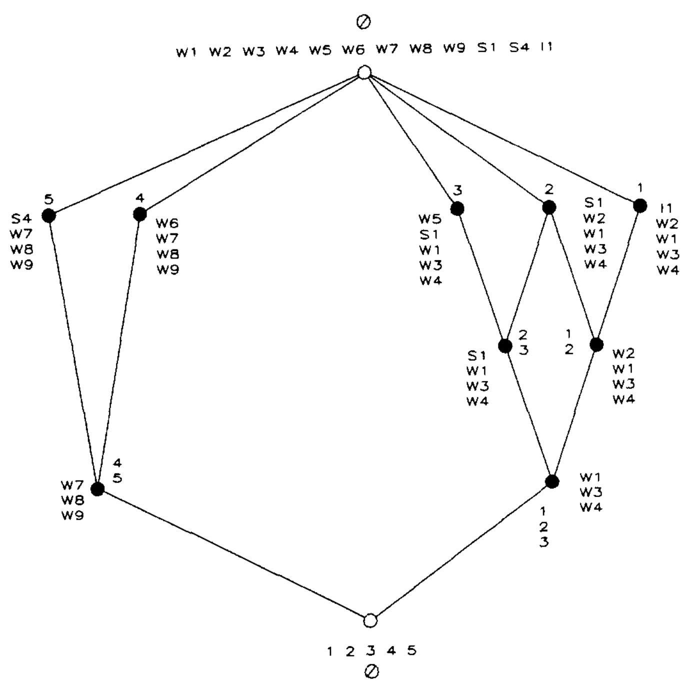

Fig. 1. Galois lattice of cliques in the Roethlisberger and Dixon (1939) data on games.

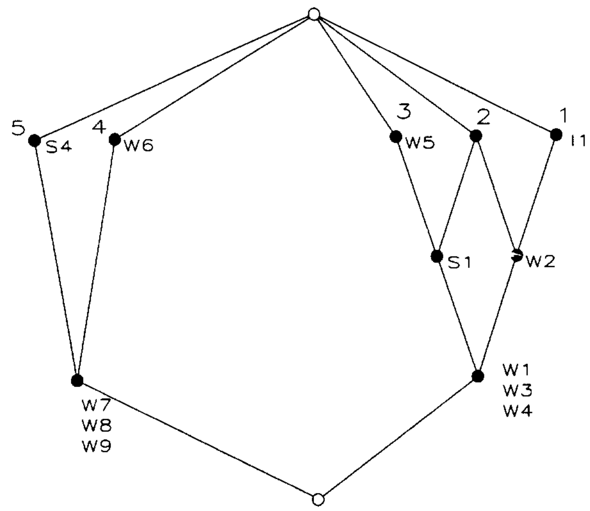

Fig. 2. The lattice of Fig. 1 with reduced labeling.

smallest element containing those actors. Similarly, each point gets the names of any cliques for which it is the greatest element containing those cliques. Points that are neither the smallest element containing any actor nor the greatest element containing any clique remain unlabeled. The clique lattice of Fig. 1 with reduced labeling is shown in Fig. 2.

Fig. 2 provides a pictorial image of the way clique memberships divide actors up into groups. It displays the relation between individuals and cliques as well as the containments both among individuals and among cliques. But it does not produce a systematic procedure that can be used to define groups. That is the problem of the next section.

5. Overlapping cliques and bounded groups

One key intuitive feature of groups is that they are generally assumed not to overlap (Freeman, 1992). This suggests a natural way of defining them in terms of the patterning of overlap among cliques. Individuals who are members of overlapping cliques can be defined as members of a common group, and individuals who are members of cliques that are distinctly separated can be defined as members of different groups. Such a definition of groups is based on establishing an external boundary that separates them from one another.

Let $C = \{c_i, c_j, \ldots\}$ be the collection of cliques. In the lattice diagram cliques are shown across the top of the picture, directly descended from the universal upper bound. I will define *overlap* o among the cliques as a binary relation in $C \times C$ that meets three conditions:

(1) $(c_i, c_i) \in o$,

(2) $c_i \cap c_j \neq \emptyset \Rightarrow (c_i, c_j) \in o$.

(3) $(c_i, c_k) \in o$ & $(c_k, c_j) \in o \Rightarrow (c_i, c_j) \in o$.

Every clique overlaps with itself. Two cliques that share one or more common members overlap. And whenever a clique overlaps with another and that other overlaps with a third, the first and the third are viewed as overlapping.

Defined in this way, overlap is reflexive, symmetric and transitive; it is an equivalence relation. Thus, the relation o partitions the set of cliques C, into subsets C', C'',... Each of the cliques in any of these subsets overlaps with at least one other clique in that subset, and no two cliques from different subsets overlap at all.

This suggests a natural way to define a bounded *group*, Γ. If A' is the union of all members of a set of overlapping cliques, C'. Then,

$$\Gamma(C') = A'.$$

A bounded group, then, is simply the union of individuals who are members of some maximal set of overlapping cliques. Since the cliques have been partitioned into non-overlapping sets on the basis of their membership, the individual members associated with these sets are partitioned also.

It is easy to see the bounded groups in the line diagram of a Galois lattice. As I indicated above, the cliques in a line diagram are all lined up at the second level of the lattice. Two cliques that overlap will be linked by descending lines that converge at some labeled point lower in the lattice. And two cliques that do not overlap will be linked only at the *unlabeled* universal lower bound. Indeed, if the universal lower bound is labeled, one or more individuals must be members of all the cliques in C and A, therefore, must be a single bounded group.

Clique 1 and clique 2 in Fig. 2, for example, share members W2, W1, W3 and W4. Cliques 1 and 3 share W1, W2 and W3 and cliques 2 and 3 share S1, W1, W3 and W4. Similarly, cliques 4 and 5 have members W7, W8 and W9 in common. But cliques 1 and 5, 2 and 5, 3 and 5, 1 and 4, 2 and 4, and 3 and 4 have *no* common members; they are distinct and non-overlapping. In Fig. 2, then, cliques 1, 2 and 3 overlap and form a single bounded group containing W1, W2, W3, W4, W5, S1 and I1. Similarly, cliques 4 and 5 overlap and form another bounded group containing W6, W7, W8, W9 and S4.

The first of these groups corresponds almost exactly to Roethlisberger and Dixon's Group A. In their original report, Roethlisberger and Dixon specified the membership of Group A as W1, W2, W3, W4, S1 and I1. W5 they described as an isolate. But the present results suggest that W5 should have been included in their Group A.

The second group uncovered here is exactly Roethlisberger and Dixon's Group B. Overall, then, the patterning of overlap among these cliques reveals a group structure that is strikingly similar to that described in the original report.

This lattice representation also suggests a natural way to define the internal structure of bounded groups. Intuitively, each individual has a position in a group, either at the core, in the periphery, or somewhere in between. What we need is a way of specifying individual positions in a lattice diagram.

Consider a lattice that has been partitioned into groups. Each group defines a sublattice consisting of the universal upper bound, the collection of overlapping cliques that make up the group, all of the elements descending from those cliques and the universal lower bound.

Except for the universal upper and lower bounds, each element in the group sublattice is linked to one or more others that fall above it by ascending lines and one or more others that fall below it by descending lines. The universal upper bound is, of course, connected to other elements only by descending lines, and the universal lower bound is connected only by ascending lines.

Now a *chain* is a sequence made up entirely of ascending lines or entirely of descending lines leading from one element to another. The *length* of that chain is the number of lines it contains.

An individual may, of course, be associated with several elements in the lattice, but for the analysis of people's positions—as in reduced labeling—we associate each individual only with the lowest element at which he or she appears.

Now consider the lengths the chains ascending from the universal lower bound to some individual in the lattice; this is the *height* of that individual. Alternatively, we can consider the lengths of the chains descending from the universal upper bound to an individual; call this the *depth* of that individual. For the moment, we will restrict our attention to those clique lattices where the height of an individual is the same along every ascending chain and that individual's depth is the same along every descending chain.

Height and depth, defined in this way, provide intuitively appealing as indexes of an individual's *position* in a group. An individual who first appears near the bottom of the lattice—one who is directly connected to the universal lower bound—is deeply embedded in that group. Such an individual may be involved in several cliques, but his or her membership in any clique is never dependent on the membership of any other individual. W1, W3 and W4 in Fig. 2, for example, are all at height = 1. They are members of all three cliques that can be reached by chains ascending from them. S1 and W2, however, first appear above W1, W3 and W4 at a height of 2. S1 and W2, therefore, are members of cliques only when W1, W3 and W4 are also members. In an important sense, then, W1, W3 and W4 are at the *core* of the group and S1 and W2 are *peripheral* to them.

W5 and I1 are, of course, even more peripheral in that same sense. W5 is involved only if W1, W3, W4 *and* S1 are all involved, and I1 is involved only if W1, W3, W4 and W2 are involved. Among the members of the first group, then, W1, W3 and W4 are at a height of 1 and are at the core, S1 and W2 are at a height of 2 and are somewhat peripheral, and W5 and I1 are only at a height of 3 and are still more peripheral. A similar treatment of the second group shows that W7, W8 and W9 at a height of 1 are core members, while W6 and S4 at a height of 2 are relatively peripheral.

These results are consistent with the descriptions reported by Roethlisberger and Dixon. They (1939, p. 464) described W3 as a leader of Group A. And they (p. 510) suggested that W2, though an affiliate of group A, was not a stable member of the group. Similarly, they (p. 509) reported that W6 participated in Group B but was not entirely accepted by its members. The present analysis yields more detail, but it does not contradict the ethnographic report.

Thus, at least in this case, examining the pattern of overlap among cliques reveals bounded groups that are internally differentiated. There are, however, circumstances in which the boundaries between the groups are less clear cut. In the next section we will take up another example and see how lattices of cliques can be used to find groups even when the boundaries are not clear.

6. Finding groups when bridges are present

Although the intuitive conception of groups stresses the idea that they do not overlap, it is generally recognized that in some cases groups may do just that. Granovetter (1973), in particular, argued for the existence of bridging ties that cut across group boundaries and link members of different groups. In this section, I will introduce a procedure that permits groups to be specified even when bridging ties are present. To do so, I will define the notion of *bridging cliques*.

A group, as defined above, is a collection of individuals who are tied together by their participation in a set of overlapping cliques. Such a group is bounded and all of the members of each of the cliques that define it are members of that group. But if two or more groups are linked by bridging ties, then there must be at least one clique that contains individuals who are members of different groups. The problem of finding bridging ties, then, is the same as the problem of finding bridging cliques.

What we need is a way to distinguish between the cliques that link the members of a single group to one another and those that serve as bridges and link members of different groups. To make such a distinction, we will have to look again at the internal structure of groups.

Intuition suggests that a person who was a member of a single group would have a clear structural position, or level, in that group—at the core, in the periphery, or somewhere in between. But if a person bridged two groups, he or she would be unlikely to find a position at exactly the same level in each of those groups.

This suggests that a clique that simply linked the members of a single group would assign individuals unambiguous positions in that clique. But a clique that bridged between two groups might include at least one individual whose position was not clearly defined.

Consider a sublattice made up of some clique c_i and all the elements descending from it. Now assume that the lattice N_5 shown in Fig. 3 is a such a clique-based

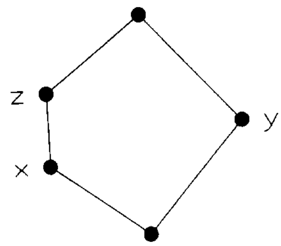

Fig. 3. The lattice *N*5.

sublattice in which the labeled points are individuals. If we evaluate this sublattice on the basis of our intuitions about individual positions, we face an ambiguity. Individuals x and y are both connected directly to the universal lower bound at the bottom (height = 1), so we must define them as core members. But, at the same time, individuals z and y are both close (depth = 1) to the clique at the top, so we must think of them as relatively peripheral members of this clique. This presents no problem so far as x and z are concerned; z is peripheral and x is at the core. But individual y is *both* peripheral and at the core; that individual is in an ambiguous position.

Let $Z = \{z_i\}$ be the collection of chains connecting the universal lower bound to the clique in a clique sublattice. Associated with each chain z_i will be a length l_i. In Fig. 3, for example, there are two such chains. The one on the right has a length of 2 while the one on the left has a length of 3. The ambiguity described above is a direct consequence of this variability in chain length. Indeed, whenever the chains connecting the universal lower bound to some clique are not homogeneous in length, that clique contains some individual who is in an ambiguous position.

A *bridging* clique, then, may be defined as one in which

$$\exists(z_i, z_j) \mid l_i \neq l_j,$$

where there is variability in the lengths of the chains connecting the clique with the universal lower bound.

When we find such bridging cliques in the lattice we can delete them. Since they are bridging, the result when they are deleted will be that some single groups will be decomposed into two or more groups that were previously linked by the bridges. Each of these new groups will be bounded.

We can see how this works by considering some data collected by Sampson (1968) in a monastery. Sampson collected these data during a period in which there was considerable conflict over the ecumenicism of Vatican II. This conflict was reflected in relationships among the novices in this monastery.

Sampson focused his attention on 18 novices. Seven of these novices (4, 5, 6, 8, 9, 10 and 11) who had arrived earlier, he defined as a group and labeled the "Loyal Opposition". Of these, he characterized 4, 6, 9 and 11 as leaders. The later arrivals, he divided into two groups. Seven (1, 2, 7, 12, 14, 15 and 16) were called

Table 3
Sampson's (1968) data on reported liking (last ranking)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	0	0	1	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0
2	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0
3	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	1
4	0	0	0	0	1	1	0	0	0	0	1	0	0	0	0	0	0	0
5	0	0	0	1	0	0	0	0	1	0	1	0	0	0	0	0	0	0
6	0	0	0	1	1	0	0	0	1	0	0	0	0	0	0	0	0	0
7	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0
8	0	0	0	1	0	1	0	0	1	0	0	0	0	0	0	0	0	0
9	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	0	0	0
10	0	0	0	1	1	0	0	0	1	0	0	0	1	0	0	0	0	0
11	0	0	0	0	1	0	0	1	0	0	0	0	0	1	0	0	0	0
12	1	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	1
14	1	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0
15	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0
16	0	1	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0
17	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
18	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0

"Young Turks" and their leaders were 1, 2 and possibly 12 (Arabie and Carroll, 1989, p. 380). And three (3, 17 and 18) were characterized as "Outcasts".

But Sampson went on to describe bridges between these groups. One novice, 13, was not clearly assignable to any group; he was a "Waverer" who drifted between and linked the Loyal Opposition and the Outcasts. Moreover, the Outcasts apparently were linked to the Young Turks. As Sampson (1968, p. 372) described them, members of these two groups were bridged by "sympathetic ties".

Sampson collected systematic data by asking each of the novices to name three others with whom they had various kinds of relationships. Specifically they were asked to rank their top three choices in terms of liking, esteem, influence, praise, disliking, negative esteem, negative influence and blame.

The vagueness and subjectivity of these questions, along with the restriction to three choices, are all unfortunate. Nevertheless, the data on who liked whom seem comparatively straightforward, and they were used for the present analysis. They are shown in Table 3. For this analysis, the rankings were collapsed by assigning 1 when any individual reported liking another at all, and 0 otherwise.

The resulting matrix is quite sparse, but more important in the present context, it is not symmetrical. It was symmetrized by taking the union, so that any pair of novices was linked if either or both of them reported liking the other.

Seventeen cliques were extracted and they produced a Galois lattice containing 45 elements. It is shown in Fig. 4.

The lattice in Fig. 4 shows that all these novices are all linked into a single bounded group. This is consistent with Sampson's report that the Outcasts and the Young Turks were linked by sympathetic ties and that novice 13 linked the Loyal

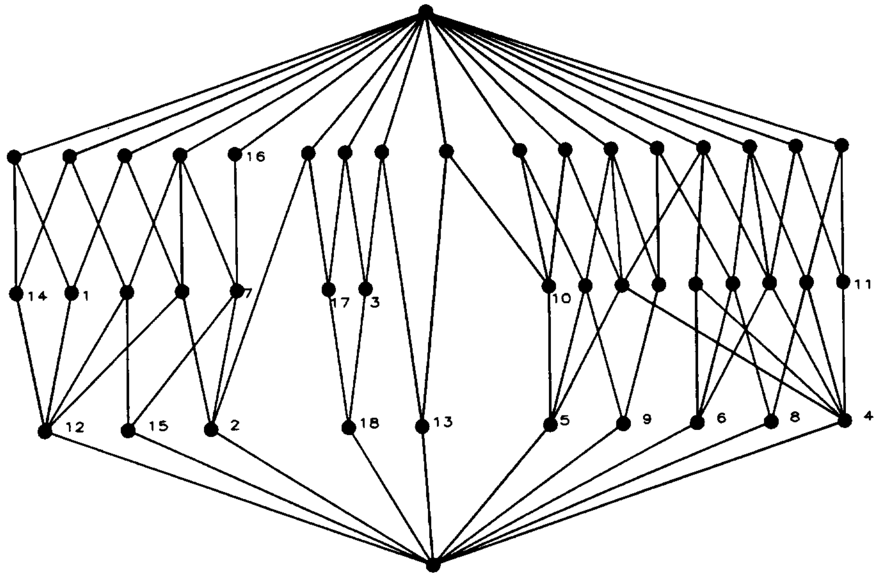

Fig. 4. Galois lattice of the Sampson (1968) data on friendship.

Opposition to the Outcasts. The problem, then, is to uncover the three groups described by Sampson in this lattice.

The lattice of Fig. 4 contains three bridging cliques as they were defined above. They involve six lines that represent bridging ties in the lattice. The new structure,

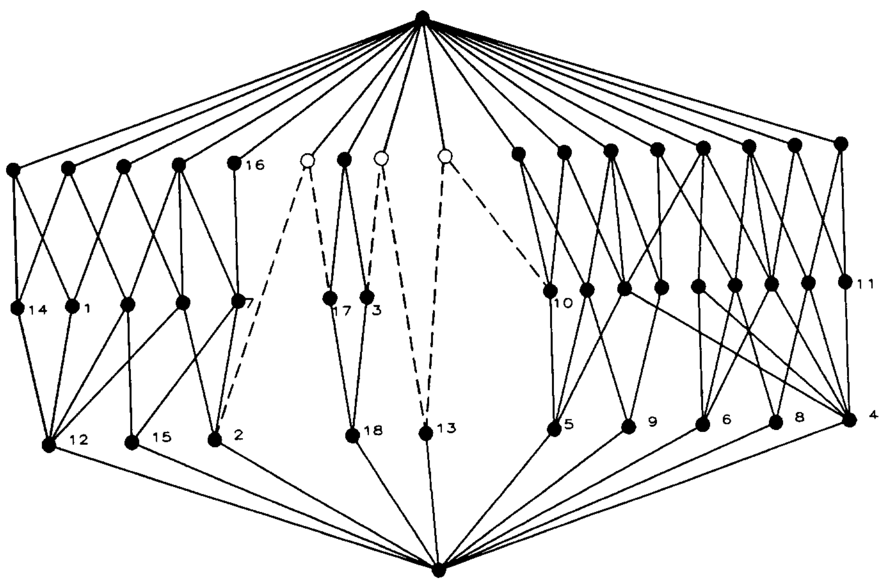

Fig. 5. The lattice of Fig. 4 showing the bridging cliques.

with bridging cliques indicated by hollow circles and bridging ties by dashed lines, is shown in Fig. 5.

According to Sampson, the Young Turks were novices 1, 2, 7, 12, 14, 15 and 16, the group on the left in the figure. The Outcasts were 3, 17 and 18, the next group. And the Loyal Opposition were 4, 5, 6, 8, 9, 10 and 11, the group on the right. Moreover, the Waverer, 13, who was described as switching back and forth between the Loyal Opposition and the Outcasts is the individual who must be dropped because he serves only as a bridge in this lattice. Overall, then, the results of this analysis of cliques yield *exactly* the group structure described by Sampson.

At the level of core and periphery the correspondence is close, but not exact. According to Sampson the leaders of the Young Turks were 1, 2 and 12. The present analysis determined that the core members were 2, 12 and 15. Sampson listed the leaders of the Loyal Opposition as 4, 6, 8 and 11. Here the core was 4, 5, 6, 8 and 9. These differ, but the agreement is still considerable.

We see, then, that bounded groups can be uncovered even in the presence of bridging ties. Thus, the Galois lattice provides a general way to use cliques to reveal groups. And the groups that are revealed embody a form that is consistent with traditional intuitive ideas about group structure. Moreover, the groups that turn up with this approach are virtually identical to those described by ethnographers. The clique formalism, it seems, produces exactly the desired result when it is used in conjunction with a Galois lattice.

7. Summary and conclusions

This paper has addressed an old problem, the difficulty of specifying exactly the conditions under which a set of individuals may be considered to be a social group. From the earliest days, sociologists have had an intuitive conception of groups. Moreover, they have had very little trouble in using their intuition to designate groups and assign individuals to them. But attempts to set down exact conditions under which a collection of individuals is or is not a group have failed. The notion of 'group' has remained an elusive concept in the field.

Here I have taken one formalism, cliques, and tied it to another, Galois lattices. Linking these two models suggested a way to define social groups based on the patterning of overlap among cliques. In addition, the structure of containment displayed in the lattice suggested a way to define people's positions in groups. These definitions were designed to embody traditional intuitive ideas about groups, and they turn out to uncover groups and positions that are consistent with ethnographic descriptions. Perhaps the most remarkable result of the present exercise is that the analyses described here are based on only the most minimal binary records of who is linked to whom.

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